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## **The four-in-a-tree problem in triangle-free graphs**

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# The four-in-a-tree problem in triangle-free graphs

Nicolas Derhy\*, Christophe Picouleau†, Nicolas Trotignon‡

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## Abstract

The three-in-a-tree algorithm of Chudnovsky and Seymour decides in time  $O(n^4)$  whether three given vertices of a graph belong to an induced tree. Here, we study four-in-a-tree for triangle-free graphs. We give a structural answer to the following question: how does look like a triangle-free graph such that no induced tree covers four given vertices ? Our main result says that any such graph must have the “same structure”, in a sense to be defined precisely, as a square or a cube.

We provide an  $O(nm)$ -time algorithm that given a triangle-free graph  $G$  together with four vertices outputs either an induced tree that contains them or a partition of  $V(G)$  certifying that no such tree exists. We prove that the problem of deciding whether there exists a tree  $T$  covering the four vertices such that at most one vertex of  $T$  has degree at least 3 is NP-complete.

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Key words: three, four, tree, algorithm, 3-in-a-tree, 4-in-a-tree, triangle-free graphs

## 1 Introduction

Many interesting classes of graphs are defined by forbidding induced subgraphs, see [2] for a survey. This is why the detection of several kinds of induced subgraphs is interesting, see [6] for a survey. In particular, the problem of deciding whether a graph  $G$  contains as an induced subgraph some graph obtained after possibly subdividing prescribed edges of a prescribed graph  $H$  has been studied. It turned out that this problem can be polynomial or NP-complete according to  $H$  and to the set of edges that can be subdivided. Details, examples and open problems are given in [6]. The most general tool for solving this kind of problems (when they are polynomials) seems to be the *three-in-a-tree* algorithm of Chudnovsky and Seymour:

**Theorem 1.1 (see [3])** *Let  $G$  be a connected graph and  $x_1, x_2, x_3$  be three distinct vertices of  $G$ . Then deciding if there exists an induced tree of  $G$  that contains  $x_1, x_2, x_3$  can be done in time  $O(n^4)$ .*

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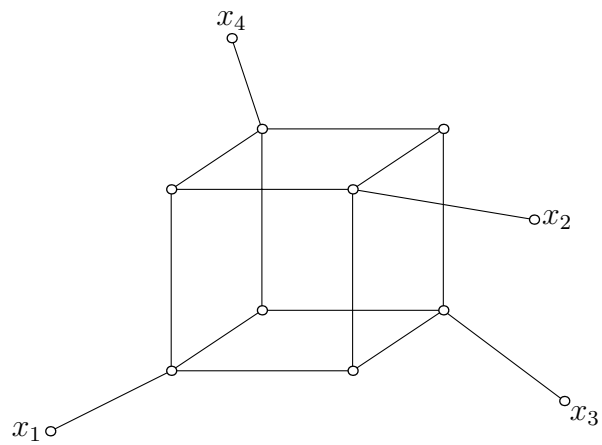


Figure 1: no tree covers  $x_1, x_2, x_3, x_4$ , first example

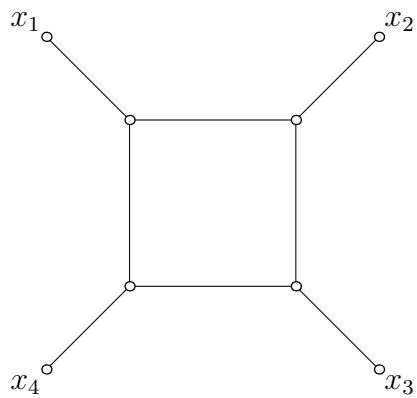


Figure 2: no tree covers  $x_1, x_2, x_3, x_4$ , second example

How to use three-in-a-tree is discussed in [3] and further evidences of its generality are given in [6]. Because of the power and deepness of three-in-a-tree, it would be interesting to generalise it. Here we study *four-in-a-tree*: the problem whose instance is a graph  $G$  together with four of its vertices, and whose question is “Does  $G$  contain an induced tree covering the four vertices?”. Since this problem seems complicated to us, we restrict ourselves to triangle-free graphs. Our approach is similar to that of Chudnovsky and Seymour for three-in-a-tree. We give a structural answer to the following question: how does look like a triangle-free graph such that no induced tree covers four given vertices  $x_1, x_2, x_3, x_4$ ? On Fig. 1 and 2, two examples of such graphs are represented. Our main result, Theorem 2.3, says that any triangle-free graph that does not contain a tree covering four vertices  $x_1, x_2, x_3, x_4$  must have the “same structure”, in a sense to be defined later, as one of the two examples. The details of the statement are given in Section 2.

Our result is algorithmic: we provide an  $O(nm)$ -time algorithm that given a graph  $G$  together with four vertices  $x_1, x_2, x_3, x_4$  outputs either an induced tree that contains  $x_1, x_2, x_3, x_4$  or a partition of  $V(G)$  certifying that no such tree exists. Note that apart from very basic subroutines such as Breadth First Search, our algorithm is self-contained. In particular it does not rely on three-in-a-tree. Our proofs will use the following result of Derhy and Picouleau:

**Theorem 1.2** (see [4]) *Let  $G$  be a triangle-free connected graph and  $x_1, x_2, x_3$  be three distinct vertices of  $G$ . Then there is an induced tree of  $G$  that contains  $x_1, x_2, x_3$ . Moreover such a tree of minimum size can be done in time  $O(m)$ .*

Note that another generalisation of three-in-a-tree would be interesting. Let us call *centered tree* any tree that contains at most one vertex of degree greater than two. Note that any minimal tree covering three vertices of a graph is centered. Hence, three-in-a-tree and three-in-a-centered-tree are in fact the same problem. So four-in-a-centered-tree is also an interesting generalisation of three-in-a-tree. But we will prove in Section 5 that it is NP-complete, even when restricted to several classes of graphs, including triangle-free graphs.

We leave open the following problems: four-in-a-tree for general graphs,  $n$ -in-a-tree for triangle-free graphs.

## Notation

All our graphs are simple and finite. We say that a graph  $G$  *contains* a graph  $H$  if  $G$  contains an induced subgraph isomorphic to  $H$ . We say that  $G$  is  $H$ -free if it does not contain  $H$ . If  $Z \subseteq V(G)$  then  $G[Z]$  denotes the subgraph of  $G$  induced by  $Z$ . When we describe the complexity of an algorithm whose input is a graph,  $n$  stands for the number of its vertices and  $m$  stands for the number of its edges.

We call *path* any connected graph with at least one vertex of degree 1 and no vertex of degree greater than 2. A path has at most two vertices of degree 1, which are the *ends* of the path. If  $a, b$  are the ends of a path  $P$  we say that  $P$  is *from  $a$  to  $b$* . The other vertices are the *interior* vertices of the path. We denote by  $v_1 - \dots - v_n$  the path whose edge set is  $\{v_1v_2, \dots, v_{n-1}v_n\}$ . When  $P$  is a path, we say that  $P$  is *a path of  $G$*  if  $P$  is an induced subgraph of  $G$ . If  $P$  is a path and if  $a, b$  are two vertices of  $P$  then we denote by  $a-P-b$  the only induced subgraph of  $P$  that is path from  $a$  to  $b$ .

Note that by *path of a graph*, we mean induced path. Also, by *tree of a graph*, we mean an induced subgraph that is a tree.

The *union* of two graphs  $G = (V, E)$  and  $G' = (V', E')$  is the graph  $G \cup G' = (V \cup V', E \cup E')$ .

A set  $X \subseteq V(G)$  is *complete* to a set  $Y \subseteq V(G)$  if there are all possible edges between  $X$  and  $Y$ . A set  $X \subseteq V(G)$  is *anticomplete* to a set  $Y \subseteq V(G)$  if there are no edges between  $X$  and  $Y$ .

When  $G$  is a graph and  $v$  a vertex,  $N(v)$  denotes the set of all the neighbors of  $v$ . If  $A \subseteq V(G)$  then  $N(A)$  denotes the set of these vertices of  $G$  that are not in  $A$  but that have neighbors in  $A$ . If  $Z \subseteq V(G)$ , then  $N_Z(A)$  denotes  $N(A) \cap Z$ . If  $H$  is an induced subgraph of  $G$ , then we write  $N_H(A)$  instead of  $N_{V(H)}(A)$ .

When we define  $k$  sets  $A_1, \dots, A_k$ , we usually denote their union by  $A$ . We use this with no explicit mention : if we define sets  $S_1, \dots, S_8$  then  $S$  will denote their union, and so on.

## 2 Main results

A *terminal* of a graph is a vertex of degree one. Given a graph  $G$  and vertices  $y_1, \dots, y_k$ , let us consider the graph  $G'$  obtained from  $G$  by adding for each  $y_i$  a new terminal  $x_i$  adjacent to  $y_i$ . It is easily seen that there exists an induced tree of  $G$  covering  $y_1, \dots, y_k$  if and only if there exists an induced tree of  $G'$  covering  $x_1, \dots, x_k$ . So, four-terminals-in-a-tree and four-vertices-in-a-tree are essentially the same problems, from an algorithmic point of view and from a structural point of view. Hence, for convenience, we may restrict ourselves to the problem four-in-a-tree where the four vertices to be covered are terminals.

As mentioned in the introduction, our main result states that a graph that does not contain a tree covering four given terminals  $x_1, x_2, x_3, x_4$  must have the “same structure” as one of the graphs represented on Fig. 1 or 2. Let us now define this precisely.

A graph that has the same structure as the graph represented on Fig 1 is what we call a *cubic structure*: a graph  $G$  is said to be a *cubic structure* with respect to a 4-tuple of distinct terminals  $(x_1, x_2, x_3, x_4)$  if there exist sets  $A_1, \dots, A_4$ ,  $B_1, \dots, B_4$ ,  $S_1, \dots, S_8$  and  $R$  such that:

1.  $A \cup B \cup S \cup R = V(G)$ ;
2.  $A_1, \dots, A_4, B_1, \dots, B_4, S_1, \dots, S_8, R$  are pairwise disjoint;
3.  $x_i \in A_i$ ,  $i = 1, \dots, 4$ ;
4.  $S_i$  is a stable set,  $i = 1, \dots, 8$ ;
5.  $S_i$  is non-empty,  $i = 1, \dots, 4$ ;
6. at most one of  $S_5, S_6, S_7, S_8$  is empty;
7.  $S_i$  is complete to  $(S_5 \cup S_6 \cup S_7 \cup S_8) \setminus S_{i+4}$ ,  $i = 1, 2, 3, 4$ ;
8.  $S_i$  is anticomplete to  $S_{i+4}$ ,  $i = 1, 2, 3, 4$ ;
9.  $S_i$  is anticomplete to  $S_j$ ,  $1 \leq i < j \leq 4$ ;

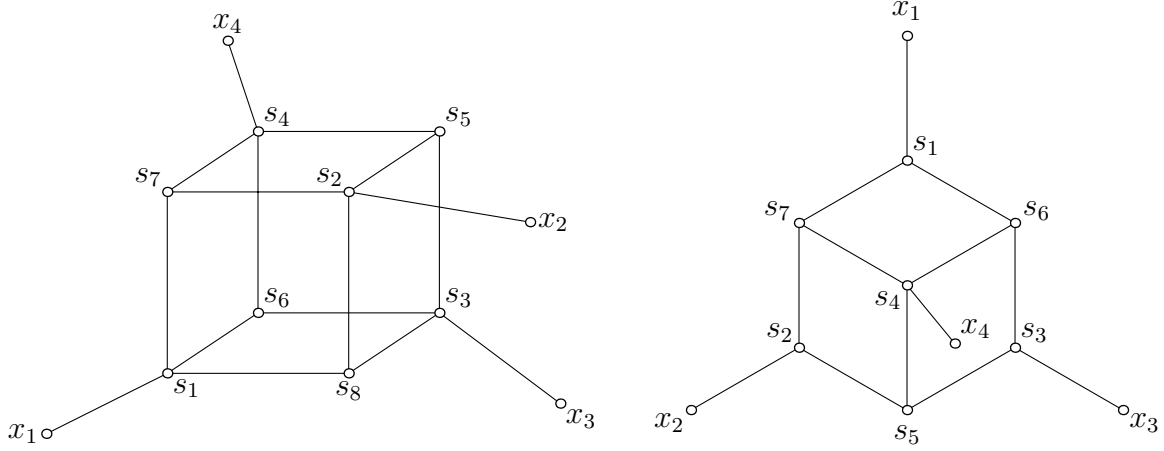


Figure 3: Two examples of cubic structures

10.  $S_i$  is anticomplete to  $S_j$ ,  $5 \leq i < j \leq 8$ ;
11.  $N(A_i) = S_i$ ,  $i = 1, 2, 3, 4$ ;
12.  $N(B_i) \subseteq S_i \cup N_S(S_i)$ ,  $i = 1, 2, 3, 4$ ;
13.  $N(R) \subseteq S_5 \cup S_6 \cup S_7 \cup S_8$ ;
14.  $G[A_i]$  is connected,  $i = 1, 2, 3, 4$ .

A 17-tuple  $(A_1, \dots, A_4, B_1, \dots, B_4, S_1, \dots, S_8, R)$  of sets like in the definition above is a *split* of the cubic structure. On Fig. 3, two cubic structures are represented. A *cubic structure of a graph  $G$*  is a subset  $Z$  of  $V(G)$  such that  $G[Z]$  is a cubic structure. The following lemma, to be proved in Section 4, shows that if a cubic structure is discovered in a triangle-free graph, then one can repeatedly add vertices to it, unless at some step a tree covering  $x_1, x_2, x_3, x_4$  is found:

**Lemma 2.1** *There is an algorithm with the following specification:*

INPUT: a triangle-free graph  $G$ , four terminals  $x_1, x_2, x_3, x_4$ , a split of a cubic structure  $Z$  of  $G$ , and a vertex  $v \notin Z$ .

OUTPUT: a tree of  $G[Z \cup \{v\}]$  that covers  $x_1, x_2, x_3, x_4$  or a split of the cubic structure  $G[Z \cup \{v\}]$ .

COMPLEXITY:  $O(m)$ .

Let us now turn our attention to our second kind of structure. A graph that has the same structure as the graph represented on Fig 2 is what we call a square structure: a graph  $G$  is said to be a *square structure* with respect to a 4-tuple  $(x_1, x_2, x_3, x_4)$  of distinct terminals if there are sets  $A_1, A_2, A_3, A_4, S_1, S_2, S_3, S_4, R$  such that:

1.  $A \cup S \cup R = V(G)$ ;

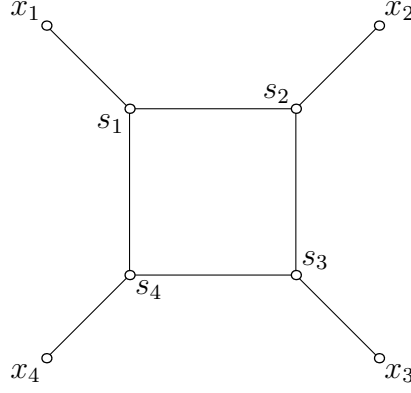


Figure 4: The smallest square structure

2.  $A_1, A_2, A_3, A_4, S_1, S_2, S_3, S_4, R$  are pairwise disjoint;
3.  $x_i \in A_i, i = 1, \dots, 4$ ;
4.  $S_i$  is a stable set,  $i = 1, \dots, 4$ ;
5.  $S_1, S_2, S_3, S_4 \neq \emptyset$ ;
6.  $S_i$  is complete to  $S_{i+1}$ , where the addition of subscripts is taken modulo 4,  $i = 1, 2, 3, 4$ ;
7.  $S_i$  is anticomplete to  $S_{i+2}, i = 1, 2$ ;
8.  $N(A_i) = S_i, i = 1, 2, 3, 4$ ;
9.  $N(R) \subseteq S_1 \cup S_2 \cup S_3 \cup S_4$ ;
10.  $G[A_i]$  is connected,  $i = 1, \dots, 4$ .

A 9-tuple  $(A_1, \dots, A_4, S_1, \dots, S_4, R)$  of sets like in the definition above is a *split* of the square structure. On Fig. 4, the smallest square structure is represented. A *square structure of a graph  $G$*  is a subset  $Z$  of  $V(G)$  such that  $G[Z]$  is a square structure. The following lemma, to be proved in Section 3, shows that if a square structure is discovered in a triangle-free graph, then one can repeatedly add vertices to it, unless at some step a cubic structure or a tree covering  $x_1, x_2, x_3, x_4$  is found:

**Lemma 2.2** *There is an algorithm with the following specification:*

INPUT: a triangle-free graph  $G$ , four terminals  $x_1, x_2, x_3, x_4$ , a split of a square structure  $Z$  of  $G$ , and a vertex  $v \notin Z$ .

OUTPUT: a tree of  $G[Z \cup \{v\}]$  that covers  $x_1, x_2, x_3, x_4$  or a split of some cubic structure of  $G$  or a split of the square structure  $G[Z \cup \{v\}]$ .

COMPLEXITY:  $O(m)$ .



From the two lemmas above the main theorem follows:

**Theorem 2.3** *Let  $G$  be a connected graph and  $x_1, x_2, x_3, x_4$  be four distinct terminals of  $G$ . Then either:*

- (1)  $G$  is a cubic or a square structure with respect to  $(x_1, x_2, x_3, x_4)$ ;
- (2)  $G$  contains a tree that covers  $x_1, x_2, x_3, x_4$ .

Moreover, exactly one these two statements (1) and (2) holds. This result is algorithmic in the sense that there exists an  $O(nm)$ -time algorithm whose input is a graph and four terminals  $x_1, x_2, x_3, x_4$  and whose output is either a partition of  $V(G)$  showing that  $G$  is a cubic or a square structure with respect to  $(x_1, x_2, x_3, x_4)$ , or a tree that covers  $x_1, x_2, x_3, x_4$ .

PROOF — Let us first check that exactly one of (1), (2) holds. This means that a square or cubic structure with respect to a 4-tuple  $(x_1, x_2, x_3, x_4)$  of terminals cannot contain a tree covering  $x_1, x_2, x_3, x_4$ . For suppose that such a tree  $T$  exists in a square structure  $G$  with a split  $(A_1, \dots, A_4, S_1, \dots, S_4, R)$ . By the definition of square structures  $T$  must contain a vertex in every  $S_i$ ,  $i = 1, 2, 3, 4$ . So,  $T$  contains a square, a contradiction.

Suppose now that such a tree  $T$  exists in a cubic structure  $G$  with a split  $(A_1, \dots, A_4, B_1, \dots, B_4, S_1, \dots, S_8, R)$ . By the definition of cubic structures  $T$  must contain a vertex in every  $S_i$ ,  $i = 1, 2, 3, 4$ . Since  $T$  contains no cycle,  $T$  has vertices in at most one of  $S_5, S_6, S_7, S_8$ , say in  $S_5$  up to symmetry. So,  $T$  contains no vertex of  $S_6 \cup S_7 \cup S_8$ . So  $x_1, x_2$  lie in two different components of  $T$ , a contradiction.

The fact that at least one of (1), (2) holds follows directly from the algorithm announced in the theorem. A description of this algorithm will complete the proof. So let us suppose that  $G$  and four terminals  $x_1, x_2, x_3, x_4$  are given. The algorithm goes through three steps:

**First step:** by Theorem 1.2 we find in time  $O(m)$  a minimal tree  $T$  that covers  $x_1, x_2, x_3$ . Note that since  $x_1, x_2, x_3$  are of degree one,  $T$  contains a vertex  $c$  of degree 3 and is the union of three paths  $P_1 = c - \dots - x_1$ ,  $P_2 = c - \dots - x_2$ ,  $P_3 = c - \dots - x_3$ .

Then we use BFS (short name for Breadth First Search, see [5]) to find a path  $Q = x_4 - \dots - w$  such that  $w$  has neighbors in  $T$ , and minimal with respect to this property. If  $w$  has a neighbor in  $P_i$  then we let  $u_i$  be the neighbor of  $w$  closest to  $x_i$  along  $P_i$ .

If  $w$  has neighbors in  $P_1, P_2, P_3$  then note that when  $1 \leq i < j \leq 3$ ,  $u_i u_j \notin E(G)$  because else,  $G$  contains a triangle. So  $V(Q \cup (u_1 - P_1 - x_1) \cup (u_2 - P_2 - x_2) \cup (u_3 - P_3 - x_3))$  induces a tree that covers  $x_1, x_2, x_3, x_4$ , so we stop the algorithm and output this tree. Note that from here on,  $wc \notin E(G)$ .

If  $w$  has neighbors in exactly one of  $P_1, P_2, P_3$ , say in  $P_1$  up to symmetry. We compute by Theorem 1.2 a tree  $T'$  of  $G[P_1 \cup \{w\}]$  that minimally covers  $w, c, x_1$ . We see that  $V(Q \cup T' \cup P_2 \cup P_3)$  induces a tree that covers  $x_1, x_2, x_3, x_4$ , so we stop the algorithm and output this tree.

So, we are left with the case where  $w$  has neighbors in two paths among  $P_1, P_2, P_3$ , say in  $P_1, P_2$  up to symmetry. Then there are two cases. First case: one of  $u_1 c, u_2 c$  is not in  $E(G)$ . Up to symmetry we suppose  $u_1 c \notin E(G)$ . We compute by Theorem 1.2 a tree  $T''$  of  $G[P_2 \cup \{w\}]$  that minimally covers  $w, c, x_2$ . We see that  $V(Q \cup (x_1 - P_1 - u_1) \cup T'' \cup P_3)$  induces a tree that covers  $x_1, x_2, x_3, x_4$ , so we stop the algorithm and output this tree. Second case:

$u_1c, u_2c$  are both in  $E(G)$ . Then we observe that  $V(P_1 \cup P_2 \cup P_3 \cup Q)$  is a square structure of  $G$ . A split can be done by putting  $A_1 = V(x_1 - P_1 - u_1) \setminus \{u_1\}$ ,  $A_2 = V(x_2 - P_2 - u_2) \setminus \{u_2\}$ ,  $A_3 = V(P_3) \setminus \{c\}$ ;  $A_4 = V(Q) \setminus \{w\}$ ,  $S_1 = \{u_1\}$ ,  $S_2 = \{u_2\}$ ,  $S_3 = \{c\}$ ,  $S_4 = \{w\}$  and  $R = \emptyset$ . We keep this square structure  $Z$  and go the next step.

**Second step:** while there exists a vertex  $v$  not in  $Z$ , we use the algorithm of Lemma 2.2 to add  $v$  to  $Z$ , keeping a square structure. If we manage to put every vertex of  $G$  in  $Z$  then we have found that  $G$  is a square structure that we output. Else, Lemma 2.2 says that at some step we have found either a tree covering  $x_1, x_2, x_3, x_4$  that we output, or a cubic structure  $Z'$ , together with a split for it. In this last case, we go the next step.

**Third step:** while there exists a vertex  $v$  not in  $Z'$ , we use the algorithm of Lemma 2.1 to add  $v$  to  $Z'$ , keeping a cubic structure. If we manage to put every vertex of  $G$  in  $Z'$  then we have found that  $G$  is a cubic structure that we output. Else, Lemma 2.1 says that at some step we have found a tree covering  $x_1, x_2, x_3, x_4$  that we output.

**Complexity analysis:** we run at most  $O(n)$  times  $O(m)$  times algorithms. So the overall complexity is  $O(nm)$ .  $\square$

### 3 Proof of Lemma 2.2

Let  $Z \subseteq V(G)$  be a square structure of  $G$  with respect to  $x_1, x_2, x_3, x_4$  together with a split like in the definition and let  $v$  be in  $V(G) \setminus Z$ . Note that  $Z$ , the split of  $Z$  and  $v$  are given by assumption.

Here below, we give a proof of the existence of the objects that the algorithm of our Lemma must output, namely a tree, a cubic structure or an augmented square structure. But this proof is in fact the description of an  $O(m)$ -time algorithm. To see this, it suffices to notice that all the proof relies on a several run of BFS or of the algorithm of Theorem 1.2, and on checks of neighborhood of several vertices. At the end, we give more information on how to transform our proof into an algorithm.

When  $s_i \in S_i \cup A_i$ , we define the path  $P_{s_i}$  to be a path from  $s_i$  to  $x_i$ , whose interior is in  $A_i$ ,  $i = 1, 2, 3, 4$ . Note that  $P_{s_i}$  exists since by Item 10 of the definition of square structures,  $G[A_i]$  is connected and by Item 8 every vertex of  $S_i$  has a neighbor in  $A_i$ ,  $i = 1, 2, 3, 4$ .

If  $v$  has no neighbor in  $A$  then  $v$  can be put in  $R$  and we obtain a split of the square structure  $Z \cup \{v\}$ . So we may assume that  $v$  has a neighbor in  $A$ , say  $a_1 \in A_1$ . We choose  $a_1$  subject to the minimality of  $P_{a_1}$ .

**Claim 3.1** *Suppose that there exists a path  $Q = v - \dots - w$  where  $Q \setminus v \subseteq R$  and such that  $w$  has neighbors in  $(A \setminus A_1) \cup (S \setminus S_1)$ . Suppose  $Q$  minimal with respect to these properties. Then either:*

1. *there exists a tree of  $G[Z \cup \{v\}]$  that covers  $x_1, x_2, x_3, x_4$ ;*
2.  *$N_S(w) = S_2 \cup S_4$  and  $N_A(w) \subseteq A_1$ ;*
3.  *$G[Z \cup \{v\}]$  contains a cubic structure.*

PROOF — Note that possibly  $Q = v = w$ . Note also that by the definition of  $R$ ,  $w$  have neighbors in  $A$  only when  $w = v$ .

(1) *If  $w$  has a neighbor in  $A_2 \cup A_4$  there exists a tree that covers  $x_1, x_2, x_3, x_4$ .*

Up to symmetry  $w$  has a neighbor  $a_2 \in A_2$ . We choose  $a_2$  subject to the minimality of  $P_{a_2}$ . Note that since no vertex of  $R$  may have a neighbor in  $A_2$ , this implies that  $Q = v = w$ .

If  $v$  has also a neighbor  $a_3 \in S_3 \cup A_3$  and a neighbor  $a_4 \in S_4 \cup A_4$  (note that  $G$  being triangle-free  $a_3 \in S_3$  and  $a_4 \in S_4$  cannot happen) then we choose  $a_3, a_4$  subject to the minimality of respectively  $P_{a_3}, P_{a_4}$ . So,  $V(P_{a_1} \cup Q \cup P_{a_2} \cup P_{a_3} \cup P_{a_4})$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . Hence we may assume that  $v$  has no neighbor in  $S_4 \cup A_4$ .

If  $v$  has a neighbor  $a_3 \in S_3 \cup A_3$  then we pick  $s_3 \in S_3$  (if  $a_3 \in S_3$ , we choose  $s_3 = a_3$ ). We let  $T_3$  be a tree of  $G[A_3 \cup \{v, s_3\}]$  that covers  $v, s_3, x_3$ . Note that  $T_3$  exists by Theorem 1.2 because  $G[A_3 \cup \{v, s_3\}]$  is connected. So,  $V(P_{a_1} \cup Q \cup P_{a_2} \cup T_3 \cup P_{s_4})$  where  $s_4 \in S_4$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . Hence we may assume that  $v$  has no neighbor in  $S_3 \cup A_3$ .

Now, we pick  $s_1 \in S_1$  and we let  $T_1$  be a tree of  $G[A_1 \cup \{v, s_1\}]$  that covers  $v, s_1, x_1$ . Note that  $T_1$  exists by Theorem 1.2 because  $G[A_1 \cup \{v, s_1\}]$  is connected. So,  $V(T_1 \cup v \cup P_{a_2} \cup P_{s_3} \cup P_{s_4})$  where  $s_3 \in S_3, s_4 \in S_4$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (1).

So, we may assume that  $w$  has no neighbor in  $A_2 \cup A_4$ .

(2) *If  $w$  has a neighbor in  $S_3$  there exists a tree that covers  $x_1, x_2, x_3, x_4$ .*

Let  $s_3$  be a neighbor of  $w$  in  $S_3$ . Note that  $G$  being triangle-free,  $w$  has no neighbor in  $S_2 \cup S_4$ . We let  $T_3$  be a tree of  $G[A_3 \cup \{w, s_3\}]$  that covers  $w, s_3, x_3$ . Note that in fact  $T_3$  is a path either from  $s_3$  to  $x_3$  or from  $w$  to  $x_3$ . So,  $V(P_{a_1} \cup Q \cup P_{s_2} \cup T_3 \cup P_{s_4})$  where  $s_2 \in S_2, s_4 \in S_4$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (2).

So, we may assume that  $w$  has no neighbor in  $S_3$ .

(3) *If  $w$  has no neighbor in  $S_2 \cup S_4$  there exists a tree that covers  $x_1, x_2, x_3, x_4$ .*

If  $w$  has no neighbor in  $S_2 \cup S_4$ , by the definition of  $Q$ ,  $w$  must have a neighbor  $a_3 \in A_3$ .

We pick  $s_3 \in S_3$ . We let  $T_3$  be a tree of  $G[A_3 \cup \{w, s_3\}]$  that covers  $w, s_3, x_3$ . So,  $V(P_{a_1} \cup Q \cup P_{s_2} \cup T_3 \cup P_{s_4})$  where  $s_2 \in S_2, s_4 \in S_4$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (3).

So, we may assume that  $w$  has a neighbor in  $S_2 \cup S_4$  (say  $s_2 \in S_2$  up to symmetry).

(4) *If  $w$  has no neighbor in  $A_3$  then either there exists a tree that covers  $x_1, x_2, x_3, x_4$  or  $N_S(w) = S_2 \cup S_4$  and  $N_A(w) \subseteq A_1$ .*

If  $s_4 \in S_4$  is a non-neighbor of  $w$ , then  $V(P_{a_1} \cup Q \cup P_{s_2} \cup P_{s_3} \cup P_{s_4})$ , where  $s_3 \in S_3$  is a tree that covers  $x_1, x_2, x_3, x_4$ . So, we may assume that  $w$  is complete to  $S_4$ . By the same way we may also assume that  $w$  is complete to  $S_2$ . Hence  $N_S(w) = S_2 \cup S_4$  and  $N_A(w) \subseteq A_1$ . This proves (4).

Note that if  $N_S(w) = S_2 \cup S_4$  and  $N_A(w) \subseteq A_1$  then the second output of our Claim 3.1 holds. So, from the definition of  $Q$  we may assume that  $w$  has a neighbor  $a_3$  in  $A_3$ . This implies  $v = w$ . We choose  $a_3$  subject to the minimality of  $P_{a_3}$ .

(5) *If  $v$  has a neighbor in  $S_4$  there exists a tree that covers  $x_1, x_2, x_3, x_4$*

Let  $s_4 \in S_4$  be such that  $vs_4 \in E(G)$ . Then  $V(P_{a_1} \cup v \cup P_{s_2} \cup P_{a_3} \cup P_{s_4})$  is a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (5).

So, we may assume that  $v$  has a non-neighbor  $s_4 \in S_4$ .

Let us finish the proof of our claim. We pick  $s_1 \in S_1$  and  $s_3 \in S_3$ . Note that  $vs_1 \notin E(G)$  since  $G$  is triangle-free. If  $a_1s_1 \notin E(G)$  then  $V(P_{a_1} \cup v \cup P_{s_2} \cup P_{a_3} \cup P_{s_4}) \cup \{s_1\}$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . So we may assume  $s_1a_1 \in E(G)$ . Symmetrically, we may assume  $s_3a_3 \in E(G)$ .

We observe that  $V(P_{a_1} \cup P_{s_2} \cup P_{a_3} \cup P_{s_4}) \cup \{s_1, s_3, v\}$  is a cubic structure. A split is given by :  $A_1 = V(P_{a_1} \setminus a_1)$ ,  $A_2 = V(P_{s_2} \setminus s_2)$ ,  $A_3 = V(P_{a_3} \setminus a_3)$ ,  $A_4 = V(P_{s_4} \setminus s_4)$ ,  $S_1 = \{a_1\}$ ,  $S_2 = \{s_2\}$ ,  $S_3 = \{a_3\}$ ,  $S_4 = \{s_4\}$ ,  $S_5 = \{s_3\}$ ,  $S_6 = \emptyset$ ,  $S_7 = \{s_1\}$ ,  $S_8 = \{v\}$ ,  $B = \emptyset$ ,  $R = \emptyset$ .  $\square$

Now, let  $C$  be the set of the  $(S_2 \cup S_4)$ -complete vertices of  $Z \cup \{v\}$ . Let  $Y$  be the set of these vertex  $w$  of  $R \cup \{v\}$  such that there exists a path from  $v$  to  $w$  whose interior is in  $R$ . Let  $Y_1$  be the set of these vertices  $w$  of  $Y \setminus C$  such that there exists a path from  $v$  to  $w$  whose interior is in  $R \setminus C$ . Let  $Y_2$  be the set of these vertices  $w$  of  $Y \cap C$  such that there exists a path from  $v$  to  $w$  whose interior is in  $R \setminus C$ . Let  $Y_3$  be  $Y \setminus (Y_1 \cup Y_2)$ . Note that  $Y = Y_1 \cup Y_2 \cup Y_3$ .

Note that we may assume that the only possible output of Claim 3.1 is  $N_S(w) = S_2 \cup S_4$  and  $N_A(w) \subseteq A_1$ . Also no vertex of  $Y$  has a neighbor in  $A_2 \cup A_3 \cup A_4$  (for  $v$  this follows from Claim 3.1, for the rest of  $Y$  this follows from the definition of  $R$ ). Note that  $v \notin Y_3$ . But  $v \in Y_2$  is possible since  $v$  can be complete to  $S_2 \cup S_4$ . So  $N_{Z \cup \{v\}}(Y_3) \subseteq Y_2 \cup S$  from the definition of  $R$ . Also  $N_{Z \cup \{v\}}(Y_2) \subseteq Y_1 \cup Y_3 \cup A_2 \cup A_4 \cup S_2 \cup S_4$ . And from Claim 3.1,  $N_{Z \cup \{v\}}(Y_1) \subseteq Y_2 \cup A_1 \cup S_1$ .

Hence, we can put all the vertices of  $Y_1$  in  $A_1$ , all the vertices of  $Y_2$  in  $S_1$  and all the vertices of  $Y_3$  in  $R$ . We obtain a split of the square structure  $Z \cup \{v\}$ .

Here is how to transform the proof above into an algorithm. We first compute  $C$ . After, we use BFS to compute  $Y$ . The output of BFS is a rooted tree whose root is  $v$ . Similarly, we compute  $Y_1, Y_2, Y_3$ . We check whether  $N_{Z \cup \{v\}}(Y_1) \subseteq Y_2 \cup A_1 \cup S_1$ . If this is true, the paragraph above shows how to output an augmented square structure. Else there is a vertex  $w \in Y_1$  such that  $w$  has neighbors in  $(A \setminus A_1) \cup (S \setminus S_1)$ . Hence by backtracking the BFS tree from  $w$ , we find a path  $Q = v \cdots w$  where  $Q \setminus v \subseteq R$  and such that  $w$  has neighbors in  $(A \setminus A_1) \cup (S \setminus S_1)$ . Moreover, the condition  $N_S(w) = S_2 \cup S_4$  and  $N_A(w) \subseteq A_1$  fails since  $w \notin C$ . So the proof of Claim 3.1 is a description of how, by just checking several neighborhoods, we can find either:

- a tree of  $G[Z \cup \{v\}]$  that covers  $x_1, x_2, x_3, x_4$  or
- a split of the a cubic structure of  $G[Z \cup \{v\}]$ ,

This completes the proof of Lemma 2.2.

## 4 Proof of Lemma 2.1

Let  $Z \subseteq V(G)$  be a cubic structure of  $G$  with respect to  $x_1, x_2, x_3, x_4$  together with a split like in the definition and let  $v$  be in  $V(G) \setminus Z$ . Note that  $Z$ , the split of  $Z$  and  $v$  are given by assumption.

Here below, we give a proof of the existence of the objects that the algorithm of our Lemma must output, namely a tree or an augmented cubic structure. But like in the proof of Lemma 2.2, this proof is in fact the description of an  $O(m)$ -time algorithm. We omit the details of how to transform the proof into an algorithm, since they are similar to these of the proof of Lemma 2.2.

When  $s_i \in S_i \cup A_i$ , we define the path  $P_{s_i}$  to be a path from  $s_i$  to  $x_i$ , whose interior is in  $A_i$ ,  $i = 1, 2, 3, 4$ . Note that  $P_{s_i}$  exists since by Item 14 of the definition of cubic structures,  $G[A_i]$  is connected and by Item 11 every vertex of  $S_i$  has a neighbor in  $A_i$ ,  $i = 1, 2, 3, 4$ .

**Claim 4.1** *The lemma holds when  $v$  has neighbors in  $A$ .*

PROOF — For suppose that  $v$  has a neighbor in  $A$ , say  $a_1 \in A_1$  (the cases with a neighbor in  $A_2, A_3, A_4$  are symmetric). We choose  $a_1$  subject to the minimality of  $P_{a_1}$ .

(1) *If there exists a path  $Q = v - \dots - w$  where  $Q \setminus v \subseteq B \cup R$  and such that  $w$  has neighbors in  $A_2 \cup A_3 \cup A_4 \cup S_2 \cup S_3 \cup S_4 \cup S_5$  then there exists a tree of  $G[Z \cup \{v\}]$  that covers  $x_1, x_2, x_3, x_4$ .*

Let  $Q$  be such a path, minimal with respect to its properties. Note that possibly  $v = w$ .

If  $w$  is adjacent to  $a_2 \in S_2 \cup A_2$ ,  $a_3 \in S_3 \cup A_3$  and  $a_4 \in S_4 \cup A_4$  then we choose  $a_2, a_3, a_4$  subject to the minimality of  $P_{a_2}, P_{a_3}, P_{a_4}$ . So,  $V(P_{a_1} \cup Q \cup P_{a_2} \cup P_{a_3} \cup P_{a_4})$  induces a tree of  $G$  that covers  $x_1, x_2, x_3, x_4$ . Hence, by symmetry, we may assume that  $w$  has no neighbor in  $S_4 \cup A_4$ .

If  $w$  is adjacent to  $a_2 \in S_2 \cup A_2$ ,  $a_3 \in S_3 \cup A_3$  then  $v = w$  because no vertex in  $B \cup R$  can have neighbors in both  $S_2 \cup A_2$ ,  $S_3 \cup A_3$  by Items 12, 13. We suppose that  $a_2, a_3$  are chosen subject to the minimality of  $P_{a_2}, P_{a_3}$ . Let  $T_2$  be a tree of  $G[A_2 \cup \{v, s_2\}]$  that covers  $x_2, v, s_2$  where  $s_2$  is some vertex of  $S_2$  (if  $a_2 \in S_2$  we choose  $s_2 = a_2$ ). Note that  $T_2$  exists by Theorem 1.2 because  $G[A_2 \cup \{v, s_2\}]$  is connected. One of  $S_6, S_7$  is non-empty by Item 6 of the definition, and we may assume  $S_7 \neq \emptyset$  because of the symmetry between  $S_2, S_7$  and  $S_3, S_6$ . So,  $V(P_{a_1} \cup Q \cup T_2 \cup P_{a_3} \cup P_{s_4}) \cup \{s_7\}$  where  $s_4 \in S_4$ ,  $s_7 \in S_7$  is a tree that covers  $x_1, x_2, x_3, x_4$ , except when  $vs_7 \in E(G)$ . But then,  $V(P_{a_1} \cup Q \cup P_{a_2} \cup P_{a_3} \cup P_{s_4}) \cup \{s_7\}$  is a tree that covers  $x_1, x_2, x_3, x_4$ , because  $a_2 \in S_2$  would entail the triangle  $a_2s_7w$ . Hence, by symmetry, we may assume that  $w$  has no neighbor in  $S_3 \cup A_3$ .

If  $w$  is adjacent to  $a_2 \in S_2 \cup A_2$  then  $a_2$  is chosen subject to the minimality of  $P_{a_2}$ . Suppose first that some vertex of  $Q$  has a neighbor  $s_6 \in S_6$ . Then  $G[A_1 \cup Q \cup S_2 \cup A_2 \cup \{s_6\}]$  is connected, so it contains a tree  $T_6$  that covers  $x_1, x_2, s_6$ . We observe that  $V(T_6 \cup P_{s_3} \cup P_{s_4})$  where  $s_3 \in S_3$ ,  $s_4 \in S_4$  is a tree that covers  $x_1, x_2, x_3, x_4$ . Hence we assume from here on that no vertex of  $Q$  has a neighbor in  $S_6$ . Let  $T_2$  be a tree of  $G[A_2 \cup \{s_2, w\}]$  that covers  $x_2, w, s_2$  where  $s_2$  is some vertex of  $S_2$  (if  $a_2 \in S_2$  we choose  $s_2 = a_2$ ). Suppose now that  $S_5 \neq \emptyset$ . We observe that  $V(P_{a_1} \cup Q \cup T_2 \cup P_{s_3} \cup P_{s_4}) \cup \{s_5\}$  where  $s_3 \in S_3$ ,  $s_4 \in S_4$ ,  $s_5 \in S_5$ , is a tree of  $G$  that covers  $x_1, x_2, x_3, x_4$  except when  $ws_5 \in E(G)$ . But in this case we observe that  $V(P_{a_1} \cup Q \cup P_{a_2} \cup P_{s_3} \cup P_{s_4}) \cup \{s_5\}$  is a tree of  $G$  that covers  $x_1, x_2, x_3, x_4$ . Hence, we may assume that  $S_5 = \emptyset$  and by Item 6 of the definition we have  $S_6, S_7, S_8 \neq \emptyset$ . If no vertex of  $Q$  has a neighbor in  $S_7 \cup S_8$  then  $V(P_{a_1} \cup Q \cup T_2 \cup P_{s_3} \cup P_{s_4}) \cup \{s_7, s_8\}$  where  $s_7 \in S_7$ ,  $s_8 \in S_8$ , is a tree of  $G$  that covers  $x_1, x_2, x_3, x_4$ . So we may assume that some vertex of  $Q$  has a neighbor in  $S_7 \cup S_8$  and we let  $u$  be the vertex of  $Q$  closest to  $v$  that has one neighbor in  $S_7 \cup S_8$ , say  $s_7 \in S_7$  (the case with one neighbor in  $S_8$  is similar because of the symmetry

between  $S_7, S_4$  and  $S_8, S_3$ ). Let  $s_2 \in S_2$ . So  $V(P_{a_1} \cup (v-Q-u) \cup P_{s_2} \cup P_{s_3} \cup P_{s_4}) \cup \{s_6, s_7\}$  is a tree of  $G$  that covers  $x_1, x_2, x_3, x_4$  except when some vertex of  $v-Q-u$  has a neighbor in  $P_{s_2}$ . But then, by the minimality of  $Q$ , we must have  $u = w$ . Now since  $G$  is triangle-free,  $a_2 \notin S_2$ . So,  $V(P_{a_1} \cup Q \cup P_{a_2} \cup P_{s_3} \cup P_{s_4}) \cup \{s_6, s_7\}$  is a tree of  $G$  that covers  $x_1, x_2, x_3, x_4$ . Hence we may assume that  $w$  has no neighbor in  $S_2 \cup A_2$ .

Now  $w$  has no neighbors in  $S_2 \cup S_3 \cup S_4 \cup A_2 \cup A_3 \cup A_4$ . So  $w$  must have a neighbor  $s_5 \in S_5$ . Hence,  $V(P_{a_1} \cup Q \cup P_{s_2} \cup P_{s_3} \cup P_{s_4}) \cup \{s_5\}$  where  $s_2 \in S_2, s_3 \in S_3, s_4 \in S_4$  is a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (1).

(2) If there exists a path  $Q = v - \dots - w$  whose interior is in  $B \cup R$  and such that  $w$  has neighbors in  $S_6 \cup S_7 \cup S_8$  then either  $Q$  contains a vertex that is complete to  $S_6 \cup S_7 \cup S_8$  or there exists a tree of  $G[Z \cup \{v\}]$  that covers  $x_1, x_2, x_3, x_4$ .

Let  $Q$  be such a minimal path. It suffices to prove that  $w$  is complete to  $S_6 \cup S_7 \cup S_8$  or that a tree covering  $x_1, x_2, x_3, x_4$  exists. By (1), we may assume that no vertex of  $Q$  has a neighbor in  $A_2 \cup A_3 \cup A_4 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ . So up to the symmetry between  $S_6, S_7, S_8$ , we may assume that  $w$  has a non-neighbor  $s_6 \in S_6$  and a neighbor  $s_7 \in S_7$  for otherwise our claim is proved. Hence  $V(P_{a_1} \cup Q \cup P_{s_2} \cup P_{s_3} \cup P_{s_4}) \cup \{s_6, s_7\}$  is a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (2).

Now, let  $C$  be the set of the  $(S_6 \cup S_7 \cup S_8)$ -complete vertices of  $Z \cup \{v\}$ . Let  $Y$  be the set of these vertex  $w$  of  $B \cup R \cup \{v\}$  such that there exists a path from  $v$  to  $w$  whose interior is in  $B \cup R$ . Let  $Y_1$  be the set of these vertices  $w$  of  $Y \setminus C$  such that there exists a path from  $v$  to  $w$  whose interior is in  $(B \cup R) \setminus C$ . Let  $Y_2$  be the set of these vertices  $w$  of  $Y \cap C$  such that there exists a path from  $v$  to  $w$  whose interior is in  $(B \cup R) \setminus C$ . Let  $Y_3$  be  $Y \setminus (Y_1 \cup Y_2)$ . Note that  $Y = Y_1 \cup Y_2 \cup Y_3$ .

By (1), we may assume that no vertex of  $Y$  has a neighbor in  $A_2 \cup A_3 \cup A_4 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ . Note that  $v \notin Y_3$ . But  $v \in Y_2$  is possible since  $v$  can be complete to  $(S_6 \cup S_7 \cup S_8)$ . So by (2),  $N_{Z \cup \{v\}}(Y_3) \subseteq Y_2 \cup S_1 \cup S_6 \cup S_7 \cup S_8$ . Also by (2),  $N_{Z \cup \{v\}}(Y_2) \subseteq Y_1 \cup Y_3 \cup A_1$ . And  $N_{Z \cup \{v\}}(Y_1) \subseteq Y_2 \cup A_1 \cup S_1$ .

Hence, we can put all the vertices of  $Y_1$  in  $A_1$ , all the vertices of  $Y_2$  in  $S_1$  and all the vertices of  $Y_3$  in  $B_1$ . We obtain a split of the cubic structure  $Z \cup \{v\}$ .  $\square$

**Claim 4.2** *The lemma holds if  $v$  is complete to  $(S_1 \cup S_2 \cup S_3 \cup S_4) \setminus S_i$ ,  $i = 1, 2, 3, 4$ .*

PROOF — We prove the claim when  $i = 4$ , the other cases are symmetric. So  $v$  is complete to  $S_1 \cup S_2 \cup S_3$ .

(1) If there exists a path  $Q = v - \dots - w$  such that  $V(Q \setminus v) \subseteq B \cup R$  and  $w$  has a neighbor  $s_4$  in  $S_4$  then  $G[Z \cup \{v\}]$  contains a tree that covers  $x_1, x_2, x_3, x_4$ .

Let us consider such a path  $Q$  minimal with respect its properties. Every vertex of  $Q \setminus v$  is in  $B_4$ . Indeed,  $B_4$  is the only set among  $B_1, \dots, B_4, R$  that allows neighbors in  $S_4$ , and there are no edges between the sets  $B_1, \dots, B_4, R$ . Hence by the properties of  $B_4$ , no vertex of  $Q \setminus v$  can have neighbors in  $S_1 \cup S_2 \cup S_3$ . So,  $V(P_{s_4} \cup Q \cup P_{s_1} \cup P_{s_2} \cup P_{s_3})$  where  $s_1 \in S_1, s_2 \in S_2, s_3 \in S_3$ , induces a tree that covers  $x_1, x_2, x_3, x_4$ . This proves (1).

Let  $Y$  be the set of these vertices  $w$  of  $B \cup R$  such that there exists a path  $Q = v - \dots - w$

whose interior is in  $B \cup R$ . If  $v$  has some neighbors in  $B_4$  then by (1) we may assume that every component of  $G[Y \cap B_4]$  contains no neighbors of vertices of  $S_4$ . So, every such component can be taken out of  $B_4$  and put in  $R$  instead. Then we may put  $v$  in  $S_8$  and we obtain a split of the cubic structure  $Z \cup \{v\}$ .  $\square$

**Claim 4.3** *The lemma holds.*

PROOF — By Claim 4.1 we may assume that  $v$  has no neighbor in  $A$ .

(1) *For the pairs  $(i, j)$  such that  $1 \leq i < j \leq 4$  and the pairs  $(i, j)$  among  $(1, 5)$ ,  $(2, 6)$ ,  $(3, 7)$ ,  $(4, 8)$  the statement below is true:*

*If there exists a path  $Q = u - \dots - w$  of  $G[B \cup R \cup \{v\}]$  such that  $u$  has a neighbor in  $S_i$  and  $w$  has a neighbor in  $S_j$  then the lemma holds.*

Let us choose such a pair  $(i, j)$  and such a path  $Q$ , subject to the minimality of  $Q$ . Note that by the definition of a cubic structure,  $V(Q) \subseteq B \cup R$  is impossible. So,  $Q$  contains  $v$ .

If  $u$  is adjacent to  $s_1 \in S_1, s_2 \in S_2, s_3 \in S_3, s_4 \in S_4$  then  $Q = u = v$  by the minimality of  $Q$ . So,  $V(P_{s_1} \cup P_{s_2} \cup P_{s_3} \cup P_{s_4} \cup Q)$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . Hence, we may assume that  $u$  (and symmetrically  $w$ ) has no neighbor in  $S_4$ .

If  $u$  is adjacent to  $s_1 \in S_1, s_2 \in S_2, s_3 \in S_3$  then  $Q = u = v$  by the minimality of  $Q$ . By Claim 4.2 we may assume that  $v$  is not complete to  $S_1 \cup S_2 \cup S_3$ , so  $v$  has a non-neighbor in  $S_1 \cup S_2 \cup S_3$ , say  $s_1 \in S_1$ . Let  $s_2 \in S_2, s_3 \in S_3$  be neighbors of  $v$ . By Item 6 of the definition, we have  $S_6 \cup S_7 \neq \emptyset$ , so up to the symmetry between  $S_2, S_7$  and  $S_3, S_6$  we may assume that there exists  $s_6 \in S_6$ . Note that  $vs_6 \notin E(G)$  because  $G$  is triangle-free. So  $V(P_{s_1} \cup P_{s_2} \cup P_{s_3} \cup P_{s_4} \cup Q) \cup \{s_6\}$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . So, we may assume that  $u$  (and symmetrically  $w$ ) has no neighbor in  $S_3$ .

If  $(i, j)$  is such that  $1 \leq i < j \leq 4$  then up to symmetry,  $u$  has a neighbor in  $s_1 \in S_1$  and  $w$  has a neighbor  $s_2 \in S_2$ . No vertex of  $Q$  has neighbors in  $S_5 \cup S_6$  because such a vertex would form a triangle or would contradict the minimality of  $Q$ . Also no vertex of  $Q$  has neighbors in  $S_3 \cup S_4$ . Indeed for  $u, w$  this follows from the preceding paragraphs, and for the interior vertices of  $Q$ , it follows from the minimality of  $Q$ . So,  $V(P_{s_1} \cup P_{s_2} \cup P_{s_3} \cup P_{s_4} \cup Q) \cup \{s\}$  where  $s \in S_5 \cup S_6$  is a tree that covers  $x_1, x_2, x_3, x_4$ .

If  $u$  has a neighbor  $s_1 \in S_1$  and  $w$  has a neighbor  $s_5 \in S_5$ , then no vertex of  $Q$  has neighbors in  $S_2 \cup S_3 \cup S_4$ . Indeed, such a vertex would form a triangle or would contradict the minimality of  $Q$ . So,  $V(P_{s_1} \cup P_{s_2} \cup P_{s_3} \cup P_{s_4} \cup Q) \cup \{s_5\}$  induces a tree that covers  $x_1, x_2, x_3, x_4$ . Similarly, we can prove that our claim holds when  $(i, j)$  is one of  $(2, 6)$ ,  $(3, 7)$ ,  $(4, 8)$ . This proves (1).

Let  $Y$  be the set of these vertex  $u$  of  $B \cup R \cup \{v\}$  such that there exists a path from  $v$  to  $u$  whose interior is in  $B \cup R$ . From (1) it follows that  $N_Z(Y)$  is included in either  $S_1 \cup N_S(S_1)$ ,  $S_2 \cup N_S(S_2)$ ,  $S_3 \cup N_S(S_3)$ ,  $S_4 \cup N_S(S_4)$  or  $S_5 \cup S_6 \cup S_7 \cup S_8$ . So, respectively to these cases,  $Y$  can be put in either  $B_1, B_2, B_3, B_4$  or  $R$ , and we obtain a split of the cubic structure  $Z \cup \{v\}$ .  $\square$

This completes the proof of Lemma 2.1.

## 5 NP-completeness of four-in-a-centered-tree

By  $K_{1,4}$  we mean the graph on five vertices with one of them of degree 4 and the others of degree 1. The NP-completeness of four-in-a-centered-tree follows directly from the fact (proved in [7] by using a refinement of a construction due to Bienstock [1]) that the problem of detecting an induced cycle passing through two prescribed vertices is NP-complete, even when restricted to  $K_{1,4}$ -free graphs. But we prefer to use the following, due to L  v  que, Lin, Maffray and Trotignon, that implies a more general result. When  $k \geq 3$ , by  $C_k$  we mean the cycle on  $k$  vertices.

**Theorem 5.1** (see [6]) *Let  $k \geq 3$  be an integer. Then the following problem is NP-complete:*

INSTANCE: *two vertices  $x, y$  of degree 2 of a graph  $G$  that does not contain  $C_3, \dots, C_k$ , except possibly  $C_6$ , and such that every vertex of  $G$  has degree 3 except  $x$  and  $y$ .*

QUESTION: *does  $G$  contain an induced cycle covering  $x, y$  ?*

We deduce easily:

**Theorem 5.2** *Let  $k \geq 3$  be an integer. Then the following problem is NP-complete:*

INSTANCE: *four terminals  $x_1, x_2, x_3, x_4$  of a graph  $G$  that does not contain  $C_3, \dots, C_k$ , except possibly  $C_6$ , and such that every vertex of  $G$  has degree 3 except  $x_1, x_2, x_3, x_4$  and one vertex of degree 4.*

QUESTION: *Does  $G$  contain a centered tree covering  $x_1, x_2, x_3, x_4$  ?*

PROOF — Let us consider an instance  $G, x, y$  of the NP-complete problem of Theorem 5.1. Let  $x', x''$  be the neighbors of  $x$  and  $y', y''$  be the neighbors of  $y$ . We prepare an instance  $G', x_1, x_2, x_3, x_4$  of our problem as follows. We delete  $x, y$ . We add five vertices  $c, x_1, x_2, x_3, x_4$  and the following edges:  $cx_1, cx_2, cx', cx'', x_3y', x_4y''$ . Now,  $G', x_1, x_2, x_3, x_4$  is an instance of our problem.

Since  $G$  contains no vertex of degree at least 4 and since  $x_1, x_2, x_3, x_4$  have degree 1, every induced centered tree of  $G'$  covering  $x_1, x_2, x_3, x_4$  must be made of four edge-disjoint paths  $c-x_1, c-x_2, c-\dots-x_3, c-\dots-x_4$ . So, such a tree exists if and only if there exists an induced cycle of  $G$  covering  $x, y$ . This proves that our problem is NP-complete.  $\square$

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